
Precision in the Perception of Direction of a Moving Pattern

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SUMMARY

This paper quantitatively examines the precision of the model of pattern motion analysis put forth by Adelson and Movshon (1982) who proposed that humans determine the direction of a moving plaid (the sum of two sinusoidal gratings of different orientations) in two steps. The velocities of the grating components are first estimated, then combined using the intersection of constraints to determine the velocity of the plaid as a whole. Under the additional assumption that the noise sources for the component velocities are independent, an approximate expression can be derived for the precision in plaid direction as a function of the precision in the speed and direction of the components. Monte Carlo simulations verify that the expression is valid to within 5% over the natural range of the parameters. The expression is then used to predict human performance based on available estimates of human precision in the judgment of single-component speed. Human performance is predicted to deteriorate by a factor of 3 as half the angle between the wavefronts (θ) decreases from 60° to 30° , but actual performance does not. The mean direction discrimination for three human observers was $4.3^\circ \pm 0.9^\circ$ (SD) for $\theta = 60^\circ$ and $5.9^\circ \pm 1.2^\circ$ for $\theta = 30^\circ$. This discrepancy can be resolved in two ways. If the noises in the internal representations of the component speeds are smaller than the available estimates or if these noises are not independent, then the psychophysical results are consistent with the Adelson-Movshon hypothesis.

INTRODUCTION

Adelson and Movshon (1982) proposed that the human visual system determines the direction of a moving pattern using a two-step procedure whereby the velocities of oriented components within the pattern are first estimated then at a later stage are combined to calculate the motion of the pattern as a whole. This hypothesis was originally formulated to explain their psychophysical findings that, in order for a plaid (the sum of two sinusoidal gratings of different orientations) to be seen to move *coherently* as a whole, the component gratings must be similar in both spatial frequency and contrast. They concluded from these psychophysical results that the human visual system analyzes plaid motion by first decomposing it into the motion of the grating components (Fig. 1). They suggested that this decomposition is the natural consequence of having orientation and spatial-frequency tuned motion sensors at the front end of the system (for a review, DeValois and DeValois, 1980). They also proposed that, at a second stage, the component velocities are recombined using the intersection of perpendicular constraints (Fennema and Thompson, 1979) to yield a measure of the motion of the

plaid as a whole. The lack of coherence for gratings of widely differing spatial frequencies was explained by assuming that the second stage of the process is only possible among sensors with similar spatial-frequency tuning.

They found support for their hypothesis in the discovery of two types of motion-sensitive neurons in the monkey visual cortex: one sensitive to component motion and one, at a higher level within the cortex, sensitive to the motion of the plaid as a whole (Movshon et al., 1986). Furthermore, a recent study has found that speed discrimination for moving plaids is consistent with the component-driven approach that Adelson and Movshon proposed (Welch, 1988).

Recently, however, there have been a number of challenges to the Adelson-Movshon hypothesis. Two studies have shown that when the gratings differ in contrast that the perceived direction of motion deviates significantly from that predicted by the intersection of constraints (Kooi, DeValois, and Wyman, 1988; Stone, Mulligan, and Watson, 1988). This problem, however, can be overcome if the perceived speed of the grating components is a function of contrast (Thompson, 1982) and perceived grating speed rather than actual grating speed is the input to the second stage of processing. In addition, even for plaids with gratings of equal contrast, certain combinations of grating speeds and orientations yield perceived directions of plaid motion significantly different from that predicted by the intersections of constraints (Ferrera and Wilson, 1988).

This study examines quantitatively the theoretical consequences of the Adelson-Movshon hypothesis on the discrimination of the direction of a moving plaid. An expression for the uncertainty in plaid direction as a function of the uncertainty in component velocity is derived assuming that the channels carrying the component information have exclusively independent sources of noise. Human performance shows that the direction of plaid motion is often determined with greater precision than is expected from the precision in independent single-component judgments within the context of the Adelson-Movshon model. These results indicate that, if the Adelson-Movshon hypothesis is correct, either the available measures of speed discrimination (McKee, Silverman, and Nakayama, 1986) are poor estimates of the internal noise associated with single-component speed or the channels carrying the component speed information share at least some of their noise sources.

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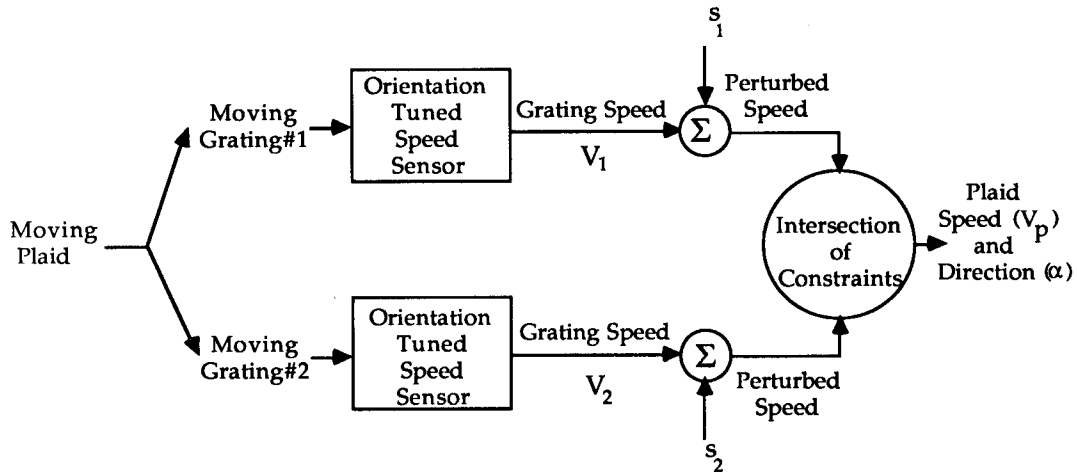


Figure 1. - Component-oriented reconstruction of pattern velocity. Adelson and Movshon (1982) proposed that plaid velocity was reconstructed from the velocity of the two grating components. The measurement of component speed is known to have some inherent noise associated with it. This noise, shown as the perturbations s_1 and s_2 , would be expected to propagate through the system and generate noise in the determination of plaid velocity. Noise in the determination of grating orientation has been neglected.

THEORY

Direction of Plaid Motion as a Function of Component Motion

The direction of motion of a plaid is a function of the relative orientation of the two grating components and their respective speeds. In the following analysis, I will determine that function. By definition, each grating component is oriented symmetrically with respect to the bisecting axis, i.e., each grating is offset equally with respect to the axis that symmetrically divides the plaid in half. The following calculation of direction is made with respect to this bisecting axis which is assumed, for simplicity, to be vertical. Figure 2 illustrates a moving plaid in velocity space with the horizontal axis indicating horizontal speed and the vertical axis indicating vertical speed. The two vectors (\vec{V}_1 and \vec{V}_2) represent the velocities of the two grating components while the vector \vec{V}_p represents the velocity of the plaid as a whole. The two components move symmetrically, θ degrees from vertical, at speeds of V_1 and V_2 , respectively, while the plaid as a whole moves at an angle α with respect to vertical, at a speed of V_p .

From figure 2, the intersection of perpendicular constraints (Fennema and Thompson, 1979) can be used to relate the motion of the plaid as a whole to the motion of the individual grating components. Specifically, plaid motion can be decomposed into component motion simply by calculating the orthogonal projections of \vec{V}_p onto the θ and $-\theta$ directions, yielding

$$V_1 = V_p \cos (\theta - \alpha) \quad (1a)$$

and

$$V_2 = V_p \cos (\theta + \alpha) \quad (1b)$$

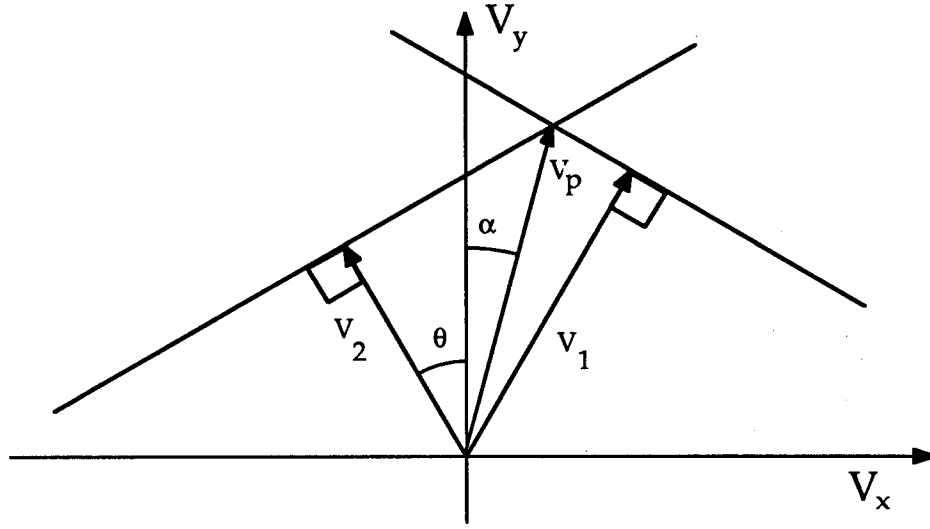


Figure 2. - Diagram illustrating plaid motion in velocity space. V_1 and V_2 are the speeds of the component gratings perpendicular to their respective wavefronts (in the directions θ and $-\theta$, respectively, with respect to the bisecting axis). V_p is the speed of the plaid as a whole; α is the direction of plaid motion with respect to the bisecting axis. The bisecting axis in this case is the vertical axis V_y .

Combining these two equations to eliminate V_p , one can show that the direction of plaid motion with respect to the bisecting axis (α) is

$$\alpha = \arctan \left[\cotan \theta \left(\frac{V_1 - V_2}{V_1 + V_2} \right) \right] \quad (2)$$

A complete derivation can be found in the appendix. Note that if $V_1 = V_2$, plaid motion is along the bisecting axis and α is zero.

Noise in the Direction of Plaid Motion Induced by Noise in Component Motion

To determine the effect of noise in component velocity on the direction of plaid motion, I first determine the effect of small perturbations in component speed (s_1 and s_2) and component direction (δ_1 and δ_2) on the direction of plaid motion (α). Just as above, plaid motion can be decomposed into component motion as follows:

$$V_1 + s_1 = V_p \cos (\theta + \delta_1 - \alpha) \quad (3a)$$

and

$$V_2 + s_2 = V_p \cos (\theta + \delta_2 + \alpha) \quad (3b)$$

where equations (3) are derived from equations (1) with s_1 , s_2 , δ_1 , and δ_2 added as small perturbations in V_1 , V_2 , and θ . Remember that each grating is oriented symmetrically with respect to the bisecting axis so θ is the same for both gratings. However, the noise in the direction of motion of the two gratings is assumed to be independent (δ_1 is independent of δ_2) as is the noise in the speeds of the two components (s_1 is independent of s_2).

Combining the two equations to eliminate V_p , one can show that the direction of plaid motion (α) is

$$\alpha = F(s_1, s_2, \delta_1, \delta_2) = \arctan \left[\frac{(V_1 + s_1) \cos (\theta + \delta_2) - (V_2 + s_2) \cos (\theta + \delta_1)}{(V_1 + s_1) \sin (\theta + \delta_2) + (V_2 + s_2) \sin (\theta + \delta_1)} \right] \quad (4)$$

Note that equation (4) reverts to equation (2), if $s_1 = s_2 = \delta_1 = \delta_2 = 0$.

The problem of determining the effect of component noise on plaid direction becomes that of determining the standard deviation of α (σ) as a function of the standard deviations of s_1 , s_2 , δ_1 , and δ_2 (σ_{s_1} , σ_{s_2} , σ_{δ_1} , σ_{δ_2}). This task can be made simpler if the nonlinear function F in equation (4) is approximated by the linear portion of its Taylor series:

$$\alpha = F(s_1, s_2, \delta_1, \delta_2) \approx F(0,0,0,0) + \frac{\partial F}{\partial s_1} s_1 + \frac{\partial F}{\partial s_2} s_2 + \frac{\partial F}{\partial \delta_1} \delta_1 + \frac{\partial F}{\partial \delta_2} \delta_2 \quad (5)$$

with $F(0,0,0,0)$ being the mean directional bias as defined in equation (2) and the remaining terms being perturbations on that mean. Since F is now a linear function, if s_1 , s_2 , δ_1 , and δ_2 are independent random variables with Gaussian, zero-mean, distributions, then the standard deviation of α is given by the following formula:

$$\sigma \approx \sqrt{\left(\frac{\partial F}{\partial s_1}\right)^2 \sigma_{s_1}^2 + \left(\frac{\partial F}{\partial s_2}\right)^2 \sigma_{s_2}^2 + \left(\frac{\partial F}{\partial \delta_1}\right)^2 \sigma_{\delta_1}^2 + \left(\frac{\partial F}{\partial \delta_2}\right)^2 \sigma_{\delta_2}^2} \quad (6)$$

Therefore, to determine σ , one must first take the partial derivatives of F (eq. 4) with respect to s_1 , s_2 , δ_1 , and δ_2 , evaluated at $(0, 0, 0, 0)$:

$$\frac{\partial F}{\partial s_1} = \frac{2 V_2 \tan \theta}{(V_1 + V_2)^2 \tan^2 \theta + (V_1 - V_2)^2} \quad (7)$$

$$\frac{\partial F}{\partial s_2} = \frac{-2 V_1 \tan \theta}{(V_1 + V_2)^2 \tan^2 \theta + (V_1 - V_2)^2} \quad (8)$$

$$\frac{\partial F}{\partial \delta_1} = \frac{V_2(V_2 - V_1) + V_2(V_1 + V_2) \tan^2 \theta}{(V_1 + V_2)^2 \tan^2 \theta + (V_1 - V_2)^2} \quad (9)$$

$$\frac{\partial F}{\partial \delta_2} = \frac{V_1(V_2 - V_1) - V_1(V_1 + V_2) \tan^2 \theta}{(V_1 + V_2)^2 \tan^2 \theta + (V_1 - V_2)^2} \quad (10)$$

Combining equations (7), (8), (9), (10) with equation (6) yields the general formula for the noise in plaid motion as a function of the noise in component motion. A complete derivation is given in the appendix.

Let us now look at the special case in which the plaid is moving straight up (i.e., $V_1 = V_2 = V$) and the noise amplitudes for each component are equal (i.e., $\sigma_s = \sigma_{s_1} = \sigma_{s_2}$ and $\sigma_\delta = \sigma_{\delta_1} = \sigma_{\delta_2}$), the general formula reduces to

$$\sigma = \frac{\sqrt{\sigma_s^2 + V^2 \tan^2 \theta} \sigma_\delta}{V \tan \theta \sqrt{2}} \quad (11)$$

Furthermore, if one assumes that the noise in V is proportional to V (McKee, 1981) and that λ is the normalized standard deviation, i.e.,

$$\sigma_s = \lambda V, \quad (12)$$

then equation 11 further reduces to

$$\sigma = \frac{\sqrt{\lambda^2 + \tan^2 \theta \sigma_\delta^2}}{\tan \theta \sqrt{2}} \quad (13)$$

One now has an equation relating the standard deviation in plaid direction to the standard deviations in component speed and direction.

Testing the Linear Approximation

Equation (13) was derived using a linear approximation (i.e., eq. 5). To verify the validity of this approximation, I compare results generated by equation (13) with those generated by Monte Carlo simulations of equation (4). The results are shown in table 1. The third column is α given by Monte Carlo simulations of equation (4) using 10,000 iterations setting the perturbations s_1 , s_2 , δ_1 , and δ_2 to random variables with independent, zero-mean, Gaussian distributions. The fourth column is α given by equation (13). Note that, although all values in this table are given in degrees, angular values for σ_δ in equations (11) and (13) must be in radians and equations (11), (13), and (14) yield values of σ in radians.

The agreement between third and fourth columns in table 1 clearly illustrates that the linear approximation was justified and that, for an extensive range of θ , λ , and σ_δ , equation (13) provides a good estimate of the actual standard deviation of α . The first three rows confirm the fact implicit in equation (13) that, if the noise in V is proportional to V , then σ is independent of V . The remaining rows show that changes in θ , λ , and σ_δ cause significant changes in α , but that agreement between the third and fourth columns is within 5% and usually better.

TABLE 1. - VALIDATION OF THE LINEAR APPROXIMATION

Conditions	Speed (V), deg/sec	Monte Carlo, deg	Equation 13, deg	% error
$\lambda = 10\%, \sigma_{\delta} = 1^{\circ}, \theta = 60^{\circ}$	0.1	2.46	2.44	0.8
	1	2.45	2.44	0.4
	10	2.48	2.44	1.6
$\lambda = 10\%, \sigma_{\delta} = 2^{\circ}, \theta = 60^{\circ}$	1	2.76	2.73	1.1
$\lambda = 10\%, \sigma_{\delta} = 10^{\circ}, \theta = 60^{\circ}$	1	7.54	7.45	1.2
$\lambda = 20\%, \sigma_{\delta} = 1^{\circ}, \theta = 60^{\circ}$	1	4.97	4.73	4.8
$\lambda = 10\%, \sigma_{\delta} = 1^{\circ}, \theta = 45^{\circ}$	1	4.10	4.11	0.2
$\lambda = 10\%, \sigma_{\delta} = 1^{\circ}, \theta = 30^{\circ}$	1	7.12	7.05	1.0

Prediction of Human Direction Discrimination

If we assume that the measured discrimination of single grating speed (McKee, Silverman, and Nakayama, 1986) and orientation (Caelli et al., 1983; Regan and Beverley, 1985) are limited by the same internal noise (λ and σ_{δ}) as the model in figure 1, then one can predict the precision in the direction of plaid motion from the precision in grating speed and direction using equation (13). To do so, however, requires knowing the precision of single-component judgments.

The best human performance in speed discrimination is around 5% (McKee, 1981). However, for a 1.5 cycle/deg sinusoidal grating moving at 1°/sec, a more appropriate value is about 11% (Welch, personal communication; McKee, Silverman, and Nakayama, 1986). Since the above values of speed discrimination are measures of the semi-interquartile difference (half the range between 25 and 75% performance), λ , the normalized standard deviation (see eq. 12), is 1.48 times larger or about 16% for the stimulus in the experiment described below.

Direction discrimination for moving random dot patterns is about 1° or 2° (DeBruyn and Orban, 1988). For moving gratings, however, direction and

orientation discrimination are confounded. Because of this, it is difficult to determine how well the direction of moving gratings might be discriminated in the absence of orientation cues. However, orientation discrimination for a 1.5 cycle/deg grating oriented 30-60° from vertical is probably 2° or higher (Caelli et al., 1983; Regan and Beverley, 1985). In any case, examination of table 1 (see second and fourth rows) shows that for values of λ of 10% (or higher) that λ dominates over values of σ_δ near 1°, i.e., doubling σ_δ produces only a 10% change in plaid direction discrimination. Therefore, the precise value of σ_δ is not critical. For simplicity, the small contribution to the uncertainty in the direction of plaid motion from σ_δ is neglected altogether and is omitted from the flow chart in figure 1. This yields the following simplified formula:

$$\sigma = \frac{\lambda \cotan \theta}{\sqrt{2}} \quad (14)$$

Equation (14) is an underestimate of the uncertainty in the direction of plaid motion because it assumes no noise in the internal representation of component direction and because it assumes that no noise is associated with the second stage of processing. Furthermore, $\lambda = 16\%$ is a conservative estimate of fractional speed discrimination in that it was measured under ideal circumstances (high contrast, long duration, large number of cycles, and highly practiced subjects) (McKee, Silverman, and Nakayama, 1986). Equation (14) also assumes that the noise sources associated with the internal representations of component speed for the two gratings are totally independent. If the noises in component speeds are correlated and proportional to speed (McKee, 1981), they will produce noise in plaid speed, but no noise in plaid direction. This can be seen graphically by noting that, if V_1 and V_2 in figure 2 are incremented or decremented by the same percentage, there is no change in plaid direction. This can also be shown analytically by noting that, if V_1 and V_2 are changed by the same proportion in equation (2), α remains unchanged. Therefore, if there is some correlation in the component noise, the predicted direction discrimination will be given by the following formula:

$$\sigma = \frac{u_0 \lambda \cotan \theta}{\sqrt{2}} \quad (15)$$

with u_0 , the fraction of the noise in the component speeds that is uncorrelated.

EXPERIMENT

Methods

To test the prediction of the previous section, human direction discrimination was measured with moving plaids. Three subjects viewed a plaid moving at $2^\circ/\text{s}$ for 300 msec in some direction between 30° to the left and 30° to the right of straight up and were asked whether the plaid moved to the right or to the left of subjective vertical. Subjects were instructed to fixate a small cross which appeared in the center of the screen 500 msec before the stimulus. The fixation point was extinguished during the stimulus presentation. The plaid stimulus was composed of two 1.5 cycle/deg sinusoidal gratings oriented symmetrically with respect to the vertical axis with θ equal to 60, 45, or 30° . The direction of plaid motion was changed by changing the relative speeds of the two grating components such that plaid speed (V_p) and orientation were always the same. The plaid image was displayed within a Gaussian window (spread = 0.25 image width) on a 19-in. monochrome monitor using an Adage RDS-300 graphics system. The stimulus was composed of superimposed halftoned gratings and was made to move using a lookup-table animation technique described in detail elsewhere (Mulligan and Stone, 1988). Subjects viewed the 5.3° by 5.3° image from a distance of 273 cm (thereby making the halftoning noise invisible) and used a chin rest to stabilize their heads. A simple updown staircase method (Cornsweet, 1962) was used to select the direction for a given trial. Two independent, randomly interleaved staircases were run for each value of θ .

When human observers were asked whether the plaid was moving to the right or to the left of the vertical axis, their performance generated a sigmoidal curve in plots of "percent of trials where the plaid is seen to move to the left" versus "actual direction of plaid motion." When the plaid moved strongly to the left, observers always responded "left." When the plaid moved strongly to the right, observers never responded "left." In between, there was a range of directions for which observers had a varying degree of uncertainty: they responded "left" a certain percentage of the time.

Human psychophysical data generated using this procedure was well fit by a cumulative Gaussian curve (the integral of a Gaussian). A simple way of quantifying human performance is therefore to determine the best fitting Gaussian (Finney, 1971). Then, one can use the mean as a measure of perceived vertical (the direction for which subjects responded "left" 50% of the time) and the standard deviation as a measure of direction discrimination (the range of directions over which there is uncertainty). The latter is plotted as σ in figure 3.

Results

Direction discrimination for moving plaids is often better than that predicted by equation (14), but is well predicted by equation (15) using the simple assumption that $u_\theta = \sin\theta$. Figure 3 shows a plot of σ for the three subjects, from equation (14), and from equation (15) for three different values of θ . At $\theta = 60^\circ$, human performance (mean $\sigma = 4.3^\circ$) is well predicted by both equation (14) (3.7°) and equation (15) (3.2°). However, all three subjects showed only a small increase in σ (decrease in performance) with increasing θ . This is as predicted by equation (15) but equation (14) erroneously predicts a threefold decrease in performance. So at $\theta = 30^\circ$, the mean human performance (5.9°) is well predicted by equation (15) (5.6°), but is on average nearly twice as good as predicted by equation (14) (11.2°).

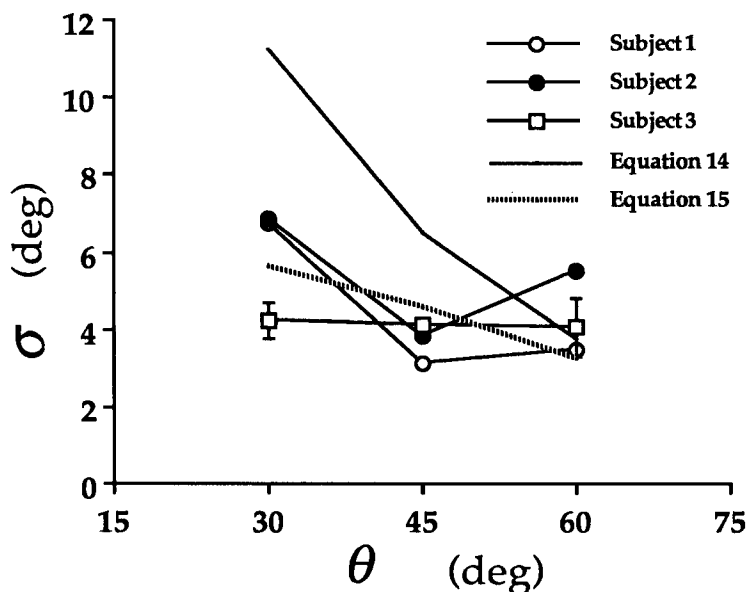


Figure 3. - Plot of the plaid direction discrimination of three subjects (1 and 2 were ignorant of the goals of this study) and two different predictions of σ versus the angle between the component wavefronts (θ). Psychometric curves were fit with a cumulative Gaussian whose standard deviation (σ) is plotted here as a measure of direction discrimination. Each data point is the mean of three runs with each run consisting of 64 observations per psychometric curve. The standard deviation for each point was always less than 1.6° and averaged 0.85° and those for subject 3 are shown explicitly. The points labelled equation 14 and 15 were evaluated using $\lambda = 16\%$.

DISCUSSION

Equation (15) can explain human direction discrimination of plaid motion if one assumes that the noises in grating speeds are correlated in relation to the angle between the gratings. Although there is no direct evidence for this type of correlation, it seems reasonable to assume that neurons whose preferred directions are close might share a significant amount of their input and therefore a significant amount of their noise. Assigning $\sin\theta$ as the fraction of noise that is uncorrelated was arbitrary. Further studies over a wider range of angles will be necessary to get a good estimate of u_θ .

The predicted curve in figure 3 generated by equation (14) is both too high and the wrong shape. It underestimates human performance and is very sensitive to the relative orientations of the gratings within the plaid (θ) whereas actual performance is not. However, an important caveat must therefore be kept in mind when interpreting this result: the predicted curve relies on the assumption that λ is 16% or higher. This assumes that the noise in the internal estimate of speed used by the second stage is expressed uncorrupted as uncertainty in judgments of single grating speed (McKee, Silverman, and Nakayama, 1986). However, it is possible that the single-component judgments made by McKee and colleagues are noisier than the internal estimates of component speed that the second stage of the model in figure 1 has access to. In other words, λ may be lower than 16%. Lower values of λ could move the entire predicted curve down below the data. Flexibility in the choice of λ can therefore make the predictions generated by equation (14) at least compatible with the psychophysical results although other noise sources at later stages would be necessary to flatten the curve.

CONCLUSION

Human discrimination of the direction of a moving plaid can be quantitatively predicted by the model proposed by Adelson and Movshon (1982), although it constrains how the model might actually be implemented with the visual cortex. The direction discrimination data presented here can be explained in at least two ways. One possibility is that the uncertainty in component speed is lower than suspected (McKee, Silverman, and Nakayama, 1986) and not rate limiting. If so, direction discrimination would be limited by some noise source at a later stage of motion processing or in the decision process. Another possibility is that the uncertainty in component speed is the rate limiting noise, but performance is enhanced and made robust to changes in θ by correlation in the noise between channels carrying information about the individual components. Both of these explanations assume that the model in figure 1 is essentially correct. Unfortunately, human direction discrimination for moving plaids does not resolve the issue of whether or not the Adelson-Movshon hypothesis is correct.

APPENDIX

1. Proof of equation (2):

From equations (1), eliminating V_p , $\frac{V_1}{\cos(\theta - \alpha)} = \frac{V_2}{\cos(\theta + \alpha)}$

then expanding the cosine terms, $\frac{V_1}{\cos\theta \cos\alpha + \sin\theta \sin\alpha} = \frac{V_2}{\cos\theta \cos\alpha - \sin\theta \sin\alpha}$

regrouping, $\cos\theta \cos\alpha \left(\frac{1}{V_1} - \frac{1}{V_2} \right) = \sin\theta \sin\alpha \left(-\frac{1}{V_1} - \frac{1}{V_2} \right)$

or $\tan\alpha = \frac{\cos\theta}{\sin\theta} \left(\frac{\frac{1}{V_1} - \frac{1}{V_2}}{-\frac{1}{V_1} - \frac{1}{V_2}} \right) = \cotan\theta \left(\frac{V_1 - V_2}{V_1 + V_2} \right)$

which immediately leads to equation (2).

2. Proof of equation (4):

Similarly, from equations (3), $\frac{V_1 + s_1}{\cos(\theta + \delta_1 - \alpha)} = \frac{V_2 + s_2}{\cos(\theta + \delta_1 + \alpha)}$

$$\frac{V_1 + s_1}{\cos(\theta + \delta_1) \cos\alpha + \sin(\theta + \delta_1) \sin\alpha} = \frac{V_2 + s_2}{\cos(\theta + \delta_2) \cos\alpha - \sin(\theta + \delta_2) \sin\alpha}$$

$$\sin\alpha \left(\frac{\sin(\theta + \delta_1)}{V_1 + s_1} + \frac{\sin(\theta + \delta_2)}{V_2 + s_2} \right) = \cos\alpha \left(\frac{\cos(\theta + \delta_2)}{V_2 + s_2} - \frac{\cos(\theta + \delta_1)}{V_1 + s_1} \right)$$

$$\tan\alpha = \frac{(V_1 + s_1) \cos(\theta + \delta_2) - (V_2 + s_2) \cos(\theta + \delta_1)}{(V_1 + s_1) \sin(\theta + \delta_2) + (V_2 + s_2) \sin(\theta + \delta_1)}$$

which immediately leads to equation (4).

3. Derivation of equation (7) and (8):

From the chain rule, $\frac{\partial \arctan(f(x))}{\partial x} = (1 + f^2(x))^{-1} \frac{\partial f(x)}{\partial x}$

so from equation (4) $\frac{\partial F}{\partial s_1} = A B$

with
$$A = \left[1 + \frac{\left((V_1 + s_1) \cos(\theta + \delta_2) - (V_2 + s_2) \cos(\theta + \delta_1) \right)^2}{\left((V_1 + s_1) \sin(\theta + \delta_2) + (V_2 + s_2) \sin(\theta + \delta_1) \right)^2} \right]^{-1}$$

and
$$B = \left[\frac{\cos(\theta + \delta_2) \left((V_1 + s_1) \sin(\theta + \delta_2) + (V_2 + s_2) \sin(\theta + \delta_1) \right) - \sin(\theta + \delta_2) \left((V_1 + s_1) \cos(\theta + \delta_2) - (V_2 + s_2) \cos(\theta + \delta_1) \right)}{\left((V_1 + s_1) \sin(\theta + \delta_2) + (V_2 + s_2) \sin(\theta + \delta_1) \right)^2} \right]$$

setting $s_1 = s_2 = \delta_1 = \delta_2 = 0$,
$$\frac{\partial F}{\partial s_1} = \left[1 + \frac{\left((V_1 - V_2) \cos \theta \right)^2}{\left((V_1 + V_2) \sin \theta \right)^2} \right]^{-1} \left[\frac{2 \cos \theta \sin \theta V_2}{\left((V_1 + V_2) \sin \theta \right)^2} \right]$$

$$\frac{\partial F}{\partial s_1} = \left[\frac{2 \cos \theta \sin \theta V_2}{(V_1 + V_2)^2 \sin^2 \theta + (V_1 - V_2)^2 \cos^2 \theta} \right]$$

and dividing through by $\cos^2 \theta$ yields equation (7).

By symmetry (i.e., exchanging the subscripts and multiplying by -1), one can generate equation (8).

4. Derivation of equations (9) and (10):

As above,
$$\frac{\partial F}{\partial \delta_1} = A B$$

with A unchanged and

$$B = \frac{(V_2 + s_2) \sin(\theta + \delta_1) \left((V_1 + s_1) \sin(\theta + \delta_2) + (V_2 + s_2) \sin(\theta + \delta_1) \right) - (V_2 + s_2) \cos(\theta + \delta_1) \left((V_1 + s_1) \cos(\theta + \delta_2) - (V_2 + s_2) \cos(\theta + \delta_1) \right)}{\left((V_1 + s_1) \sin(\theta + \delta_2) + (V_2 + s_2) \sin(\theta + \delta_1) \right)^2}$$

setting $s_1 = s_2 = \delta_1 = \delta_2 = 0$,
$$\frac{\partial F}{\partial \delta_1} = \left[1 + \frac{\left((V_1 - V_2) \cos \theta \right)^2}{\left((V_1 + V_2) \sin \theta \right)^2} \right]^{-1} \left[\frac{\sin^2 \theta V_2 (V_1 + V_2) - \cos^2 \theta V_2 (V_1 - V_2)}{\left((V_1 + V_2) \sin \theta \right)^2} \right]$$

$$\frac{\partial F}{\partial \delta_1} = \left[\frac{\sin^2 \theta V_2 (V_1 + V_2) - \cos^2 \theta V_2 (V_1 - V_2)}{(V_1 + V_2)^2 \sin^2 \theta + (V_1 - V_2)^2 \cos^2 \theta} \right]$$

Dividing by $\cos^2 \theta$ yields equation (9); exchanging subscripts and multiplying by -1 yields equation (10).

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